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A SUBMODEL OF THE ROTATIONAL MOTIONS OF A GAS IN A UNIFORM FORCE FIELD†

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An invariant submodel, constructed using a subalgebra of the sum of the rotation, time transfer and Galilean transfer is considered within the framework of the Podmodeli program [1]. A group classification is constructed and simple solutions are obtained. The submodel is reduced to symmetrical form. Assertions are made on the hyperbolicity, characteristics and force discontinuities. The necessary conditions for solutions to exist without discontinuities on the axis of symmetry are derived and their asymptotic submodel is investigated. © 1998 Elsevier Science Ltd. All rights reserved.

1. THE SUBMODEL EQUATIONS AND CHARACTERISTICS

The equations of gas dynamics allow of an 11-parameter continuous group of transformations with Lie algebra L_{11} [1]. A class of group solutions corresponds to each subalgebra. After all dissimilar subalgebras were enumerated, a study of the properties of the group solutions began [2–6]. Third-rank invariant submodels correspond to one-dimensional subalgebras. The submodel of rotational motions of a gas in a uniform force field is determined by the invariant solutions of the one-dimensional algebras $H = \{\beta X_4 + X_7 + \beta X_{10}\}$, $\beta \neq 0$, where $X_4 = t\partial_x = \partial_u$ is the operator of Galilean transfer with respect to x, $X_7 = \partial_\theta$ is the operator of rotation around the x axis and $X_{10} = \partial_t$ is the operator of time transfer.

Cylindrical coordinates x, r, θ are used. The projections of the velocity vector onto the corresponding cylindrical unit vectors are denoted by U, V and W. The representation of the invariant solution has the form

$$U = t + u(q, r, s),\ V = \upsilon(q, r, s),\ W = \beta^{-1}r(1 + w(q, r, s))$$
$$\rho = \rho(q, r, s),\ p = p(q, r, s);\ q = x - t^2/2,\ s = \beta\theta - t \tag{1.1}$$

The equations of the submodel are obtained by substituting representation (1.1) into the equations of gas dynamics

$$\rho D\mathbf{u} + (p_q, p_r, \beta^2 r^{-2} p_s) = \rho\mathbf{a},\quad A^{-1}Dp + \operatorname{div}\mathbf{u} = -r^{-1}\upsilon$$
$$D\rho + \rho\operatorname{div}\mathbf{u} = -r^{-1}\rho\upsilon \quad \text{or} \quad DS = 0 \tag{1.2}$$

where

$$D = u\partial_q + \upsilon\partial_r + w\partial_s,\quad \mathbf{u} = (u, \upsilon, w),\quad \mathbf{a} = (-1, \beta^{-2}r(1+w)^2, -2r^{-1}\upsilon(1+w))$$
$$\operatorname{div}\mathbf{u} = u_q + \upsilon_r + w_s,\quad A = \rho c^2,\quad c^2 = \partial f/\partial\rho$$

$p = f(\rho, S)$ is the equation of state, p is the pressure, ρ is the density and S is the entropy.

System (1.2) can be reduced to symmetrical form by a linear replacement of the velocities

$$\upsilon^i = b^i_j u^j (u^1 = u,\ u^2 = \upsilon,\ u^3 = w),\quad u^j = c^j_i \upsilon^i,\quad b^i_j c^j_k = \delta^i_k$$

After these replacements its matrix form is

$$D^i \mathbf{f}_{x^i} = \mathbf{D},\quad \mathbf{f} = (\upsilon^1, \upsilon^2, \upsilon^3, p, S)^T \tag{1.3}$$

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$$\mathbf{D} = (d^1, d^2, d^3, d^4, 0)^T, \quad d^4 = -\upsilon^i (r^{-1} c_i^2 + c^j_{i x^j})$$

$$d^i = \rho b_k^i a^k + \rho b^i_{j x^n} c_k^n \upsilon^k c_m^j \upsilon^m, \quad x^1 = q, \quad x^2 = r, \quad x^3 = s$$

$$B^i = \begin{Vmatrix} \rho c_k^i \upsilon^k & 0 & 0 & b_i^1 g^i & 0 \\ 0 & \rho c_k^i \upsilon^k & 0 & b_i^2 g^i & 0 \\ 0 & 0 & \rho c_k^i \upsilon^k & b_i^3 g^i & 0 \\ c_1^i & c_2^i & c_3^i & A^{-1} c_k^i \upsilon^k & 0 \\ 0 & 0 & 0 & 0 & c_k^i \upsilon^k \end{Vmatrix}, \quad i = 1, 2, 3$$

$$g^1 = g^2 = 1, \quad g^3 = r^{-2}\beta^2$$

For the matrices B^i to be symmetrical the following six equations must be satisfied

$$c_j^1 = b_1^j, \quad c_j^2 = b_2^j, \quad c_j^3 = b_3^j \beta^2 r^{-2} \tag{1.4}$$

to determine the nine elements of the matrix $C = (c_j^i)$.

Suppose $\mathbf{c}^i = (c_1^i, c_2^i, c_3^i)^T$. Then $|\mathbf{c}^i|^2 = g^i$, $\mathbf{c}^i \cdot \mathbf{c}^j = 0$ and the matrix C is determined, apart from the rotation, if we specify the direction of one of the vectors $\mathbf{c}^i$.

For example, suppose $\mathbf{c}^3 = (0, 0, \beta r^{-1})^T$. Then

$$\mathbf{c}^1 = (\cos\alpha, \sin\alpha, 0)^T, \quad \mathbf{c}^2 = (\cos\alpha, -\sin\alpha, 0)^T$$

$$\mathbf{b}_1 = (\cos\alpha, \sin\alpha, 0), \quad \mathbf{b}_2 = (-\sin\alpha, \cos\alpha, 0), \quad \mathbf{b}_3 = (0, 0, \beta^{-1} r)$$

Here

$$d^1 = \rho(-\cos\alpha + \beta^{-2} r(1+w)^2 \sin\alpha), \quad d^2 = \rho(\sin\alpha + \beta^{-2} r (1+w)^2 \cos\alpha)$$

$$d^3 = -\beta^{-1} \upsilon \rho (2 + w), \quad d^4 = -r^{-1} \upsilon \rho$$

The eigenvalues of the matrices B^i are

$$\lambda_1^i = u^i, \quad \lambda_{2,3}^i = \rho u^i, \quad \lambda_{4,5}^i = \frac{1}{2} u^i \left[\rho + A^{-1} \pm ((\rho - A^{-1})^2 + 4 g^i (u^i)^{-2})^{1/2}\right]$$

Hence, the matrix B^i is positive definite if

$$\rho > 0; \quad u^i > c \ (i = 1, 2), \quad u^3 > \beta c r^{-1} \ (i = 3)$$

System (1.2) can have three invariant characteristics [7]

$$C_0: \ u h_q + \upsilon h_r + w h_s = 0 \quad \text{(triple entropy)} \tag{1.5}$$

$$C_\pm: \ (u h_q + \upsilon h_r + w h_s)^2 - c^2 (h_q^2 + h_r^2 + \beta^2 r^{-2} h_s^2) = 0 \tag{1.6}$$

For real characteristics to exist the quadrature form on the left-hand side of (1.6)

$$Q = \xi U \xi^T, \quad U = c^{-2} \mathbf{u} \otimes \mathbf{u} - \operatorname{diag}(1, 1, \beta^2 r^{-2})$$

must be of alternating sign. In this case system (1.2) is hyperbolic.

Theorem 1.1. For a fixed solution the region in which system (1.2) is hyperbolic is given by the inequality

$$u^2 + \upsilon^2 + \beta^{-2} r^2 w^2 > c^2 \tag{1.7}$$

Proof. The eigenvalues of the form Q are found from the equation

$$f(\lambda) = \lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0$$

$$J_1 = c^{-2} |\mathbf{u}|^2 - 2 - \beta^2 r^{-2}, \quad J_2 = 1 + 2\beta^2 r^{-2} - c^{-2}(u^2 + v^2)(1 + \beta^2 r^{-2}) - 2c^2 w^2$$

$$J_3 = \beta^{-2} r^{-2} c^{-2}(u^2 + v^2 - c^2) + c^{-2} w^2$$

By Routh's theorem [8, p. 475], the number χ of positive roots of the polynomial $f(\lambda)$ is equal to the number of sign changes in the series of expressions

$$1, -J_1, J_2 - J_3 J_1^{-1}, -J_3$$

Hence we have the conditions which define the region of hyperbolicity. When $\chi = 1$ three cases are possible ($a = c^{-2}(u^2 + v^2) > 0, b = c^{-2}w^2 > 0$)

1) $J_1 > 0,\ J_2 J_1 < J_3,\ J_3 > 0 \Rightarrow a + b > 2 + \beta^2 r^{-2}$

2) $J_1 < 0,\ J_2 J_1 > J_3,\ J_3 > 0 \Rightarrow a + b > 1 + \beta^2 r^{-2}$

$(1 + \beta^2 r^{-2})(a - 2) + 2b < 0$

3) $J_1 < 0,\ \ J_2 J_1 < J_3,\ \ J_3 > 0 \Rightarrow \{a + b < 2 + \beta^2 r^{-2}, (1 + \beta^2 r^{-2})(a - 2) +$

$+2b > 0\} \cup \{a + b < 1 + \beta^2 r^{-2}, \beta^2 r^{-2} b + a > 1\}$

By combining these regions we obtain the statement of the theorem.

When $\chi = 2$ three cases of contradictory inequalities are possible

1) $J_1 > 0,\ J_2 J_1 > J_3,\ J_3 < 0$

2) $J_1 > 0,\ J_2 J_1 < J_3,\ J_3 < 0$

3) $J_1 < 0,\ J_2 J_1 > J_3,\ J_3 < 0$

Note. In physical variables inequality (1.7) takes the form

$$U^2 + V^2 + W^2 - 2tU - 2\beta^{-1} rW + t^2 + \beta^{-2} r^2 > c^2$$

The region defined by this inequality does not coincide with the region of supersonic flow. For sufficiently large r, system (1.2) is hyperbolic in any solution defined for these r, if $w \neq 0$.

The normal to the invariant surface is given by the formula

$$\mathbf{n} = (h_q^2 + h_r^2 + h_s^2)^{-1/2}(h_q, h_r, h_s)$$

The characteristics have the form

$$C_0\colon\ \mathbf{u} \cdot \mathbf{n} = u_n = 0 \tag{1.8}$$

$$C_\pm\colon\ \ u_n = \pm c(h_q^2 + h_r^2 + h_s^2)^{-1/2}(h_q^2 + h_r^2 + \beta^2 r^{-2} h_s^2)^{1/2} \tag{1.9}$$

The projection of the velocity vector onto the normal of the characteristics $C_\pm$ depends on the position of the point on the surface. When $r = \beta$ the projection is identical with the velocity of sound c, when $r > \beta$ it is less than c, and when $r < \beta$ it is greater than c.

The conditions on the characteristics [9, p. 54] are

$(D = u\partial_q + v\partial_r + w\partial_s)$:

C_0: $DS = 0$

$$h_q^{-1}(\rho Du + p_q + 1) = h_r^{-1}(\rho Dv + p_r - \beta^{-2} r(1 + w)^2) =$$

$$= h_s^{-1}(\rho D(\beta^{-1} rw) + \beta r^{-1} p_s - \beta^{-1} v(2 + w)) \quad (h = h^0) \tag{1.10}$$

$$C_\pm:\ h_q(\rho Du+p_q+1)+h_r(\rho Du+p_r-\beta^{-2}r(1+w)^2)+$$
$$+r^{-1}h_s(\rho D(rw)+\beta^2r^{-1}p_s+\upsilon(w+2))=\pm\rho c(h_q^2+h_r^2+h_s^2)^{1/2}\times$$
$$\times(h_q^2+h_r^2+\beta^2r^{-2}h_s^2)^{-1/2}(u_q+\upsilon_r+w_s+A^{-1}Dp+r^{-1}\upsilon)\ (h=h^\pm) \tag{1.11}$$

System (1.8)–(1.11) of eight equations for the functions $h^0, h^\pm, u, \upsilon, w, p, S$, together with the equation of state, is the characteristic form of Eqs (1.2).

2. THE STREAMLINE AND INTEGRALS

The streamline for submodel (1.2) is specified by the following system of equations

$$L:\ dq/u=dr/\upsilon=ds/w$$

The conditions on the characteristic C_0 give two integrals of motion along the streamline: the entropy integral

$$S(q,r,s)=S(L)$$

and the Bernoulli integral

$$u^2+\upsilon^2+\beta^{-2}r^2w^2+I(c^2)+2q-\beta^{-2}r^2=C(L) \tag{2.1}$$

where $I(c^2) = 2\int c^2\rho^{-1}d\rho$ is a single-valued increasing function and such that $I\to 0$ as $c^2\to 0$ and $I\to\infty$ as $\rho\to 0$ [9, p. 101].

The following critical velocity $c_*(r, q, L)$ is defined

$$c_*^2+I(c_*^2)=C+r^2-2q$$

The region of hyperbolicity of the system can be established by comparing the expression $\omega^2 = u^2 + \upsilon^2 + \beta^{-2}r^2w^2$ with the critical velocity. If $\omega < c$ we have

$$\omega^2+I(\omega^2)<\omega^2+I(c^2)=C+r^2-2q=c_*^2+I(c_*^2)<c^2+I(c^2)$$

Then $\omega < c_* < c$.

If $\omega > c$, similar discussions lead to the relation $\omega > c_* > c$.

A stream tube consists of streamlines passing through the initial disc K_R of radius R, perpendicular to the vector $\mathbf{u}$ at the centre of the disc. Suppose K is another section of the tube, while Σ is the side surface of the body T_R, formed from the stream tube by two sections K_R and K. On Σ the velocity $\mathbf{u}$ is perpendicular to the normal $\mathbf{n}$. Integration of the third equation of system (1.2) over the body leads to the law of conservation of flow rate along the stream tube

$$\int_{K_R} r\rho\mathbf{u}\cdot\mathbf{n}d\sigma=\int_K r\rho\mathbf{u}\cdot\mathbf{n}d\sigma=Q(T_R)$$

In the limit as $R\to 0$ we obtain the integral along the streamline

$$r\rho|\mathbf{u}|F=Q(L)$$

where $F = \lim KK_R^{-1}$ as $R\to 0$.

3. THE EQUATIONS OF THE FORCE DISCONTINUITY

The invariant surface of the force discontinuity $h(q, r, s)$ has a velocity in the direction of the normal

$$D_n=(th_q+h_s)(h_q^2+h_r^2+\beta^2r^{-2}h_s^2)^{-1/2}$$

We can write the relative velocity in terms of the invariants

$$v_n = u_n - D_n = (uh_q + vh_r + wh_s)(h_q^2 + h_r^2 + \beta^2 r^{-2} h_s^2)^{-1/2}$$

The contact discontinuity [9, p. 38] is given by the equations

$$[p] = p_2 - p_1 = 0, \;\; u_i h_q + v_i h_r + w_i h_s = 0, \;\; i = 1, 2$$

where the subscript i denotes the values of quantities on different sides of the force discontinuity.

The shock-wave equations have the form

$$[\mathbf{u}_\sigma] = 0, \;\; [\rho v_n] = 0, \;\; [p + \rho v_n^2] = 0, \;\; [v_n^2 + I] = 0$$

where $\mathbf{u}_\sigma$ is the tangential component of the velocity vector. The last equation is equivalent to the Hugoniot condition. The first equation defines h for specified velocity jumps

$$h_q^{-1}[u] = h_r^{-1}[v] = \beta^{-2} r^2 h_s^{-1}[w]$$

Zemplen's theorem holds for the submodel considered. For state 1 $|v_{n1}| > c_1$ in front of the shock-wave front, and for state 2 $|v_{n2}| > c_2$ behind the shock-wave front.

4. GROUP CLASSIFICATION OF THE SUBMODEL

System (1.2) with arbitrary elements $A = A(p, \rho)$, β allows of the following equivalence conversion

$$q' = a_1^2 q, \;\; r' = a_1^2 r, \;\; u' = a_1 u, \;\; v' = a_1 v, \;\; \rho' = a_2 \rho, \;\; p' = a_1^2 a_2 (p + a_3)$$
$$A' = a_1^2 a_2 A, \;\; s' = a_1 s, \;\; \beta' = a_1 \beta$$

By choosing the parameter a_1 we can make $\beta' = 1$. Hence, the parameter β of submodel (1.2) is unimportant.

The result of the group classification of system (1.2) is shown in Table 1, while N is the number of the expansion from [1, Table 1]. The nucleus $N = 1$ occurs in all five Lie algebras, and R is the dimensionality of the algebra. All the algebras are a factor in the normalizer algebras of subalgebra H, in the corresponding algebras with special functions A [1, Table 1], with respect to H. The functions g, encountered in Table 1, are arbitrary.

Table 1

N	A	Operators	r
1	$g(\rho, p)$	$\{Y_1 = \partial_q, Y_2 = \partial_s\}$	2
3	$pg(\rho p^{-1})$	Z_p	3
9	$g(\rho)$	Z_1	3
11	p	Z_p, Z_1	4
13	0	$Z_{g(\rho)} = \rho g'(\rho)\partial_\rho + g(\rho)\partial_p$	∞

Table 2

r	i	Basis	Nor	Subalgebras from L_{11}	Subalgebras from [1, Table 6]	Aut L_{11}
2	1	1,2	=2.1	1,7,4+10	3.9^0	
1	1	2+α1	2.1	α1−4−10,7+β(4+10)	~2.7	Γ
1	2	1	2.1	1,7+β(4+10)	~2.11	A_{11}

The kernel of the allowed algebras is an Abel algebra. Hence, the optimum system of subalgebras is obvious, since there are no internal automorphisms. The progress of the subalgebras into the algebra L_{11} and the establishment of similarity ($\sim$) with the subalgebras of [1, Table 6] is important. This correspondence is given in Table 2 using conversions from Aut L_{11} [1, Table 3]. In Table 2 we show the number of subalgebras from the corresponding tables and a number of operators. In the column headed Nor we show the number (r, i) of the subalgebra, which is a normalizer.

5. SIMPLE SOLUTIONS

The invariant solution in the kernel is specified by functions that are independent of q and s. System (1.2) can then be integrated $u = u_0 + D^{-1}\int \rho r dr$, $\upsilon = Dr^{-1}\rho^{-1}$, $w = Br^{-2} - 1$, $S = S_0$, $p = f(\rho)$.

We have

$$D^2 r^{-2}\rho^{-2} + B^2\beta^{-2}r^{-2} + I(\rho) = C^2, \quad I(\rho) = 2\int_0^\rho \rho^{-1} f'(\rho)d\rho \geqslant 0 \tag{5.1}$$

where u_0, S_0, D, B, C are constants.

Theorem 5.1. For a normal gas Eq. (5.1) defines a two-valued function $\rho(r)$, defined in the interval $r \geqslant r_0 > 0$. One of the branches $\rho > \rho(r_0) = \rho_0$ increases monotonically, and the radial velocity for this is subsonic. The other branch $\rho < \rho_0$ decreases monotonically and the radial velocity for this is supersonic.

Proof. For a normal gas the following relations hold [9, p. 101]

$$I \to 0 \text{ as } \rho \to 0; \quad I \to \infty \text{ as } \rho \to \infty; \quad I_\rho > 0, \quad f' = c^2, \quad f'' > 0$$

Equation (5.1) defines a bounded function, since, as $\rho \to \infty$, the equation is not satisfied. As $r \to \infty$ we have two limits: a non-zero limit $\rho \to \rho_1 > 0$, $I(\rho_1) = C^2$ and a zero limit $r\rho \to DC^{-1}$. Thus, (5.1) defines a bounded two-valued function. As $r \to 0$ and for bounded values of ρ, Eq. (5.1) is not satisfied. The branches $\rho(r)$ are then defined in the set $r \geqslant r_0 > 0$.

The quantity r_0^2 can be obtained as the minimum of the function

$$r^2 = R(\rho) = (D^2\rho^{-2} + B^2\beta^{-2})(C^2 - I(\rho))^{-1}$$

We have

$$R'(\rho) = 2(C^2 - I(\rho))^{-2} D^2\rho^{-3}[F(\rho) - C^2], \quad F(\rho) = I(\rho) + f'(\rho)(1 + B^2\rho^2\beta^{-2}D^{-2})$$

Since $F' > 0$, a unique root ρ_0 of the equation $F(\rho) = C^2$ exists. Then $r_0^2 = R(\rho_0)$.

When $\rho > \rho_0$, $r > r_0$ we have $f'(\rho) > f'(\rho_0)$, $\rho r > \rho_0 r_0$. Then

$$c^2 - \upsilon^2 = f'(\rho) - D^2 r^{-2}\rho^2 > f'(\rho_0) - D^2 r_0^{-2}\rho_0^{-2} = 0 \quad \text{and} \quad R'(\rho) > 0, \quad \upsilon < c$$

When $\rho < \rho_0$, $r > r_0$ we have

$$c^2 - \upsilon^2 < D^2(r_0^{-2}\rho_0^{-2} - r^{-2}\rho^{-2}) = D^2[I(\rho) - I(\rho_0) + B^2\beta^{-2}(r^{-2} - r_0^{-2})] < 0$$

and $R'(\rho) < 0$, $\upsilon > c$.

In physical variables the solution is given by the formulae

$$U = t + D^{-1}\int_{r_0}^{r} \rho r dr + u_0, \quad V = Dr^{-1}\rho^{-1}, \quad W = Br^{-1}$$

It defines the flow of gas from a cylindrical source $r \leqslant r_0$ with a twist $W \neq 0$ and a constant acceleration along the x axis.

For many submodels isobaric flows give a considerable class of exact solutions [2]. This is not the case in the submodel considered.

Theorem 5.2. Isobaric flows of submodel (1.2) are specified by the formulae

$$u = s + a(r), \quad v = 0, \quad w = -1, \quad \rho = \rho(r, q + s^2/2 + sa(r)), \quad p = p_0 \tag{5.2}$$

Proof. When $p = p_0$ system (1.2) becomes overdetermined

$$\begin{gathered} Du = -1, \quad Dv = \beta^{-2} r(1+w)^2, \quad Dw = -2r^{-1} v(1+w), \quad D\rho = 0 \\ u_q + v_r + w_s + v\, r^{-1} = 0 \quad (D = u\partial_q + v\partial_r + w\partial_s) \end{gathered} \tag{5.3}$$

Along the streamline L: $u^{-1}dq = v^{-1}dr = w^{-1}ds = d\tau$ the system has four integrals

$$\rho = \rho(L), \quad r^2(1+w) = f(L), \quad v^2 + \beta^{-2} r^2 (1+w)^2 = g^2(L), \quad u + \tau = h(L)$$

From the equations for a streamline we obtain three other integrals

$$\begin{gathered} q - \tau^2/2 - \tau u = h_1(L), \quad rv - \tau v^2 - \beta^{-2} \tau r^2 (1+w)^2 = g_1(L) \\ s + \tau - \beta \operatorname{arctg}(\beta v r^{-1} (1+w)^{-1}) = f_1(L) \end{gathered}$$

Eliminating the parameter τ we obtain six integrals along the streamline

$$\begin{gathered} \rho = a_1, \quad r^2(1+w) = a_2, \quad v^2 + \beta^{-2} r^2 (1+w)^2 = a_3 \\ \frac{r(1+w)\operatorname{tg}((u-s)\beta^{-1}) + v\beta}{r(1+w) - \beta v \operatorname{tg}((u-s)\beta^{-1})} = a_4, \quad 2q + u^2 = a_5 \\ rv + uv^2 + \beta^{-2} r^2 u (1+w)^2 = a_6 \end{gathered} \tag{5.4}$$

If all a_i are constants, the equations of system (1.2) are contradictory. Suppose $a_i(\lambda)$ and assume that $s = s(q, r, \lambda)$ in integrals (5.4). The equality $(a_3 r^2 - a_2^2 \beta^{-2})^{1/2} + (a_5 - 2q)^{1/2} a_3 = a_6$, which connects the independent variables λ, q, r, follows from (5.4). It is satisfied identically only when $a_2 = a_3 = a_6 = 0$. Then, $v = 0$ and $w = 1$, and (5.2) follows from (5.3).

In physical variables we obtain the solution

$$\begin{gathered} U = \beta\theta + a(r), \quad V = 0, \quad W = 0, \, p = p_0 \\ \rho = \rho(r, x + (\beta\theta - t)\, a(r) + \beta\theta(\beta\theta/2 - t)) \end{gathered}$$

which describes the non-isentropic flow round a plane wedge $\theta_1 < \theta < \theta_2$ by a rectilinear steady flow of particles moving parallel to the x axis (the generatrix of the wedge). Here the function $a(r)$ must be monotonic and is defined for all values of r, while the density distribution over r is arbitrary and time-dependent.

6. THE NECESSARY CONDITIONS FOR A SOLUTION TO EXIST WITHOUT A SINGULARITY ON THE AXIS

Submodel (1.2) may have a singularity when $r = 0$. The solution without singularities is represented by a series in negative powers of the variable r, if the series r (summation is over integers $k \geqslant 0$)

$$\begin{gathered} u = \sum u_k r^k, \quad v = r \sum v_k r^k, \quad w = \sum w_k r^k, \quad \rho = \sum \rho_k r^k \\ p = P(q) + r^2 \sum p_k r^k, \quad A = \sum A_k r^k, \quad A_k = (k!)^{-1} D_r^k A(p, \rho)\,|_{r=0} = \\ = A_\rho^0 \rho_k + A_p^0 p_{k-2} + A_{\rho\rho}^0 \rho_1 \rho_{k-1} + A_{p\rho}^0 \rho_1 p_{k-3} + \ldots, \quad A_\rho^0 = A_\rho(P, \rho_0), \ldots \end{gathered} \tag{6.1}$$

Quantities with a zero subscript satisfy the equations

$$D_0u_0 = -\rho_0^{-1}P' - 1, \quad D_0v_0 = -v_0^2 + \beta^{-2}(1+w_0)^2 - 2\rho_0^{-1}p_0$$
$$D_0w_0 + \beta^2\rho_0^{-1}p_{0s} = -2v_0(1+w_0), \quad D_0\rho_0 = A_0^{-1}\rho_0u_0P' \tag{6.2}$$
$$u_{0q} + w_{0s} = -2v_0 - A_0^{-1}u_0P', \quad D_0 = u_0\partial_q + w_0\partial_s$$

This system is of the Cauchy type in the variable s if $w_0 \neq 0$. The quantities $P'(q), A_0 = A(P, \rho_0)$ and β are arbitrary elements of system (6.2).

The quantities $u_k, v_k, w_k, \rho_k, p_k, k > 0$ are found from the linear system of equations

$$D_0u_k = -(kv_0 + u_{0q})u_k - u_{0s}w_k + P'\rho_0^{-2}\rho_k + g_1^k$$
$$D_0v_k = -v_{0q}u_k - (k+2)v_0v_k + (2\beta^{-2}(1+w_0) - v_{0s})w_k + 2p_0\rho_0^{-2}\rho_k - (k+2)p_k + g_2^k$$
$$D_0w_k + \beta^2\rho_0^{-1}p_{ks} = -w_{0q}u_k - 2(1+w_0)v_k - (w_{0s} + (k+2)v_0)w_k + \beta^2\rho_0^{-2}p_{0s}\rho_k + g_3^k \tag{6.3}$$
$$D_0\rho_k = (\rho_0A_0^{-1}P' - \rho_{0q})u_k - \rho_{0s}w_k + (-kv_0 + A_0^{-1}u_0P' - \rho_0u_0A_0^{-2}A_\rho^0P')\rho_k + g_4^k$$
$$u_{kq} + w_{ks} = -A_0^{-1}P'u_k - (k+2)v_k + u_0A_0^{-2}A_\rho^0P'\rho_k + g_5^k$$

where g_i^k are expressed in terms of $u_j, v_j, w_j, \rho_j, p_{j-1}, (j = 0, \ldots, k-1)$.

The systems of equations (6.3) are Cauchy type systems in the variable s if $w_0 \neq 0$. Hence, the formal series (6.1) can be constructed. These series specify the necessary asymptotic behaviour of the solution without a singularity close to the $r = 0$ axis.

Series (6.1) can also be constructed for invariant ∂_s-solutions. In this case system (6.2) has integrals when $u_0 \neq 0$

$$A(P,\rho_0)d\rho_0 = \rho_0 dP \Rightarrow S(P,\rho_0) = S_0, \quad \text{or} \quad \rho_0 = G(P)$$
$$w_0 = -1 + Cu_0\rho_0 \tag{6.4}$$
$$u_0^2 + I(P) = D - 2q, \quad I = 2\int G^{-1}(P)dP > 0$$

The remaining equations define

$$v_0 = -(u_{0q} + u_0\rho_0^{-1}\rho_{0q})/2, \quad p_0 = \rho_0(C^2\beta^{-2}u_0^2\rho_0^2 - v_0^2 - u_0v_{0q})/2$$

All the quantities with a zero subscript are defined, apart from an arbitrary function $P(q)$ and constants S_0, C and D.

Quantities with the subscript k are found from (6.3). The first two equations give v_k and p_k. The remaining equations form a Cauchy type system for finding u_k, w_k, ρ_k.

Thus, the formally invariant ∂_s-solution is determined with an arbitrariness in three constants and one function.

It follows from integral (6.4) that a solution is only possible when $q \leq D/2$. When $q = D/2$ the solution describes a state of vacuum. Hence, the ∂_s-solution in the region of the $r = 0$ axis describes flow into a vacuum.

When $P'' = 0$ we can consider the invariant ∂_q-solution. System (6.2) can be integrated

$$P = P_0, \quad \rho_0 = R$$
$$w_0 = \frac{1}{4}C\beta^2 + B\sin(2\beta^{-1}s + \Phi), \quad v_0 = -\beta^{-1}B\cos(2\beta^{-1}s + \Phi)$$
$$u_0 = -\int w_0^{-1}ds + D, \quad p_0 = \frac{1}{2}R\beta^{-2}\left(\left(1 + \frac{1}{4}C\beta^2\right)^2 - B^2\right)$$

where P_0, R, C, B, D, Φ are constants.

In physical variables the solution defines flow that is π-periodic in θ with a pressure that is independent of θ, x, t close to the $r = 0$ axis.

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